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Fuller Phenomenon in optimal control problems

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Optimal control problem

$$\int_0^T (\varphi_0(x) + u\varphi_1(x))dt \to \min, \qquad \dot{x} = f_0(x) + uf_1(x)$$

$$x(0) \in B_0 \subset \mathbb{R}^n, \qquad x(T) \in B_T \subset \mathbb{R}^n$$

$$|u| \leq 1$$

Here x is a state variable, u is a scalar control,
$\varphi_i : \mathbb{R}^n \to \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1$,
the functions φ_i, f_i are smooth enough,
B_0, B_T are smooth manifolds.
The admissible controls $u(t)$ need to be measurable,
the admissible trajectories $x(t)$ are assumed to be absolutely continuous.

Pontryagin's maximum principle

Define the Hamiltonian

$$H = H_0(x, \psi) + uH_1(x, \psi),$$

where $H_0(x, \psi) = f_0(x)\psi - \frac{1}{2}\varphi_0(x)$,
$H_1(x, \psi) = f_1(x)\psi - \frac{1}{2}\varphi_1(x)$.
We have the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \psi}, \qquad \dot{\psi} = -\frac{\partial H}{\partial x} \tag{1}$$

and

$$H\left(x\left(t\right), \psi\left(t\right), u^{opt}\left(t\right)\right) = \max_{0 \le u \le 1} H\left(x\left(t\right), \psi\left(t\right), u\right) \tag{2}$$

Singular extremal

Since the Hamiltonian H is linear in u, hence to maximize it over the interval $u \in [-1, 1]$ we need to use boundary values depending on the sign of $H_1 = \psi$.
The maximum condition yields:

$$u = +1 \text{ for } H_1 > 0, \quad u = -1 \text{ for } H_1 < 0.$$

An extremal $(x(t), \psi(t)),\ t \in (t_0, t_1)$, is called ***singular*** if $H_1(x(t), \psi(t)) = 0$ for $t \in (t_0, t_1)$.
To find the control on singular extremal $(x(t), \psi(t))$ one needs to differentiate the identity $H_1(x(t), \psi(t)) = 0$.

Order of a singular extremal

We say that a number q is an ***order*** *of a singular trajectory* iff

$$\frac{\partial}{\partial u}\frac{d^k}{dt^k}\bigg|_{(1)} H_1(x,\psi) = 0, \qquad k = 0,\dots,2q-1,$$

$$\frac{\partial}{\partial u}\frac{d^{2q}}{dt^{2q}}\bigg|_{(1)} H_1(x,\psi) \neq 0$$

in some open neighborhood of the singular trajectory $(x(t),\psi(t))$.
It is known that for optimal trajectories a singular arc of even order is joined with a chattering trajectory.
A ***chattering trajectory*** is a trajectory with infinite number of control switchings in a finite time interval.

Fuller problem

Minimize

$$\int_0^\infty s^2(t)dt \tag{3}$$

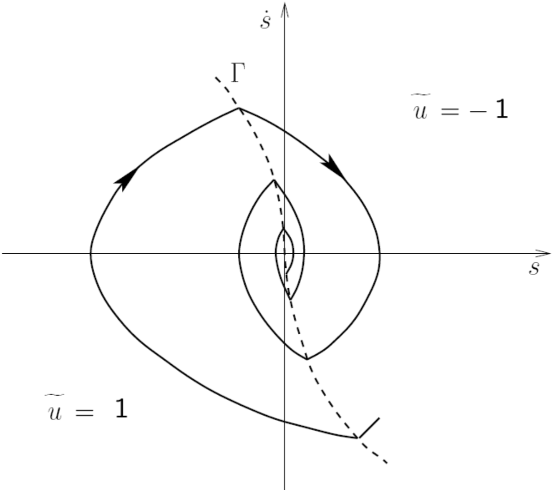
subject to

$$\ddot{s}(t) = u(t), \qquad -1 \leq u(t) \leq 1$$

with initial conditions

$$s(0) = a, \quad \dot{s}(0) = b \tag{4}$$

Optimal feedback control



Optimal Solutions in Fuller Problem

Denote $\dot{s} = v$

- The curve
$$s = -Cv^2 sgn v$$
is the optimal switching set of the Fuller Problem. Here $C \approx 0,444623\ldots$.
- Twisting around the origin the optimal trajectories attain the origin in a finite time and intersect the switching curve at a countable set of points.
- The optimal control equals 1 from the left-hand side of the switching curve and equals -1 from the right-hand side of it.

Optimality conditions for the Fuller problem

$$H(s,v,\phi,\psi) = -\frac{1}{2}s^2 + v\phi + u\psi = H_0 + uH_1$$

Let $(\widetilde{s}(t), \widetilde{v}(t), \widetilde{u}(t))$ be an optimal solution in the problem. Then there exist continuous functions $\phi(t), \psi(t)$ such that

$$\begin{aligned} \dot{\phi} &= -\frac{\partial H}{\partial s} = s, \\ \dot{\psi} &= -\frac{\partial H}{\partial v} = -\phi, \end{aligned}$$

$\widetilde{u} = +1$ for $\psi > 0$ and $\widetilde{u} = -1$ for $\psi < 0$.

If $\psi = 0$ for $t \in (t_0, t_1)$ then an extremal

$$(s(t), v(t), \phi(t), \psi(t)), \quad t \in (t_0, t_1),$$

is a ***singular*** one.

Singular Control

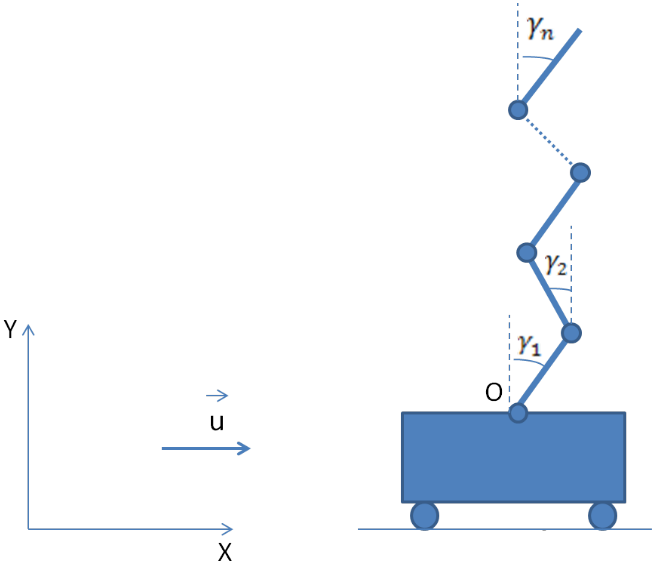
Denote $z = (s, v, \phi, \psi)$. We have:

$$H_1(z(t)) = \psi(t) \equiv 0, \quad \frac{d}{dt} H_1(z(t)) = 0 \Rightarrow -\phi(t) = 0$$

$$\begin{aligned} \frac{d^2}{dt^2} H_1(z(t)) &= 0 \Rightarrow -s(t) = 0, \\ \frac{d^3}{dt^3} H_1(z(t)) &= 0 \Rightarrow -v(t) = 0, \\ \frac{d^4}{dt^4} H_1(z(t)) &= 0 \Rightarrow -u(t) = 0. \end{aligned} \tag{5}$$

The singular extremal in the Fuller problem $s = 0$, $v = 0$.

n -link inverted pendulum



n -link inverted pendulum

M is the cart mass, s is the cart position,
g is the acceleration of gravity, u is the force applied to the cart,
γ_i is the angle of deviation of the ith link from the vertical line,
m_i is the mass of the ith link,
r_i is the distance from the lower end of the ith link to its center of mass,
I_i is the moment of inertia with respect to the center of mass of the ith link,
and l_i is the length of the ith link $(i = 1, \dots, n)$.

n -link inverted pendulum. Motion equations

The equations of motion are

$$a_{11}\ddot{s} + \sum_{i=1}^{n} a_{1,i+1}\ddot{\gamma}_i \cos\gamma_i - \sum_{i=1}^{n} a_{1,i+1}\dot{\gamma}_i^{\,2} \sin\gamma_i = u$$

$$a_{1,i+1}\ddot{s}\cos\gamma_i + a_{i+1,i+1}\ddot{\gamma}_i + \sum_{j=1}^{n} a_{i+1,j+1}\ddot{\gamma}_j \cos(\gamma_i + \gamma_j) - \qquad (6)$$

$$-\sum_{j=1}^{n} a_{i+1,j+1}\dot{\gamma}_j^2 \sin(\gamma_i + \gamma_j) - b_i \sin\gamma_i = 0, \qquad i = 1, \ldots, n.$$

n -link inverted pendulum.

We assume that the initial state of the system is in a suciently small neighbourhood of the upper unstable equilibrium position

$$\gamma_1 = \dot{\gamma}_1 = \cdots = \gamma_n = \dot{\gamma}_n \equiv 0. \tag{7}$$

We study the problem of stabilization of the pendulum in the neighbourhood of position (7) in the sense of minimization of the quadratic functional

$$\int_0^\infty \langle \gamma, \gamma \rangle \, dt \to \min, \tag{8}$$

Linearized model. Optimal control problem

$$\int_0^\infty \langle Kx(t), x(t)\rangle \, dt \to \min \tag{9}$$

on the trajectories of the system

$$\ddot{x}(t) - \Lambda x(t) = lu(t) \tag{10}$$

with the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = y_0. \tag{11}$$

Here, the control $u(t)$ is a bounded scalar function:

$$|u(t)| \leq 1, \tag{12}$$

Optimal control problem. Notation

$x \in \mathbb{R}^n$ are phase variables,
I is the vector consisting of 1's,
K is a constant symmetric positive definite $n \times n$ matrix,
Λ is a constant diagonal positive definite $n \times n$ matrix,
$\Lambda = \text{diag}\,\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_1, \ldots, \lambda_n > 0$.

Pontryagin Maximum Principle

$$H(x, y, \phi, \psi) = -\frac{1}{2}\langle Kx, x\rangle + \langle y, \phi\rangle + \langle \Lambda x, \psi\rangle + \langle I, \psi\rangle u$$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \Lambda x + Iu \\ \dot{\phi} &= Kx - \Lambda\psi \\ \dot{\psi} &= -\phi \end{aligned} \tag{13}$$

$$u(t) = sgn\, H_1(t) = sgn\, \langle I, \psi(t)\rangle \tag{14}$$

Singular solution

$$H_1(t) = \langle l, \psi(t) \rangle \equiv 0, \quad \frac{dH_1}{dt} = -\langle l, \phi \rangle,$$

$$\frac{d^2 H_1}{dt^2} = -\langle l, Kx - \Lambda\psi \rangle, \quad \frac{d^3 H_1}{dt^3} = -\langle l, Ky + \Lambda\phi \rangle,$$

$$\frac{d^4 H_1}{dt^4} = -\langle l, K(\Lambda x + lu) + \Lambda(Kx - \Lambda\psi) \rangle = \tag{15}$$

$$= -\langle l, (K\Lambda + \Lambda K)x \rangle + \langle l, \Lambda^2\psi \rangle - u\langle Kl, l \rangle$$

Singular control

$$S = \{ \langle l, \phi \rangle = 0, \ \langle l, Kx - \Lambda\psi \rangle = 0, \\ \langle l, Ky + \Lambda\phi \rangle = 0, \ -\langle l, (K\Lambda + \Lambda K)\, x \rangle + \langle l, \Lambda^2 \psi \rangle = 0 \}$$

$$u_{oc}(t) = \frac{-\langle l, (K\Lambda + \Lambda K)\, x(t) \rangle + \langle l, \Lambda^2 \psi(t) \rangle}{\sigma}$$

Here $\sigma = \langle Kl, l \rangle$.Since $|u(t)| \leqslant 1$ we consider the domain

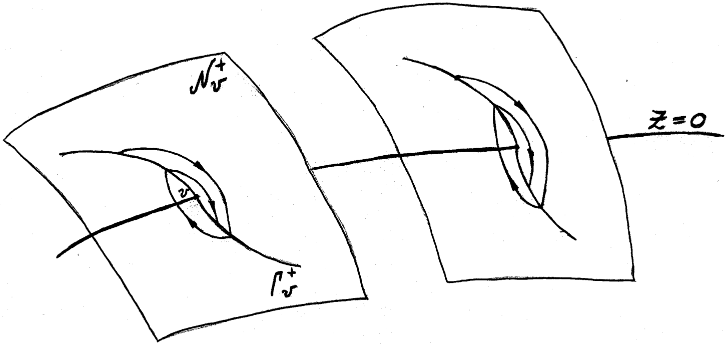
$$\left| -\langle l, (K\Lambda + \Lambda K)\, x \rangle + \langle l, \Lambda^2 \psi \rangle \right| \leq \sigma$$

Hamiltonian system in the singular surface S

$$
\begin{cases}
\dot{x}_k = y_k, \\
\dot{y}_k = \lambda_k x_k + \frac{1}{\sigma}\left(-\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}(\lambda_i + \lambda_j - (\lambda_1 + \lambda_2))x_i + \right. \\
\qquad\qquad \left. + \sum_{i=3}^{n} (\lambda_1 - \lambda_i)(\lambda_2 - \lambda_i)\psi_i \right) \\
\qquad\qquad\qquad\qquad k = 1, \dots, n; \\
\\
\dot{\psi}_k = -\phi_k, \\
\dot{\phi}_k = \sum_{j=1}^{n} k_{kj} x_j - \lambda_k \psi_k, \qquad k = 3, \dots, n;
\end{cases}
$$

singular surface

Optimal solution reaches S in a finite time with an infinite number of control switchings (chattering regime). Then the motion proceeds along the singular surface and asymptotically approaching the origin.



Model problem with two-dimensional control

$$\int_0^{\infty} (x_1^2(t) + x_2^2(t))\,dt \to \min$$

$$\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2$$

$$\dot{y}_1 = u_1, \quad \dot{y}_2 = u_2$$

$$x_i(0) = s_i^0, \; y_i(0) = {r_i}^0,$$

$$i = 1, 2$$

$$u_1^2 + u_2^2 \leq 1$$

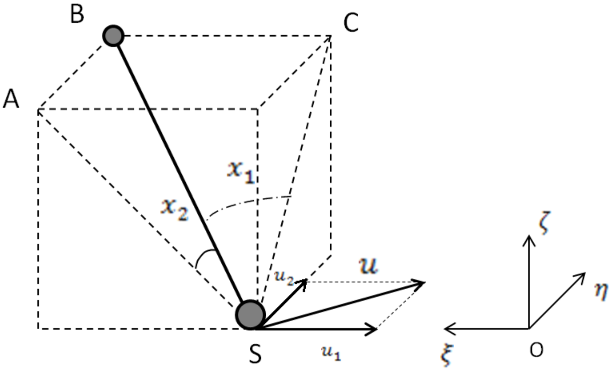
Optimal Solutions of the Model Problem

- Optimal solutions, starting from a small enough neigbourhood of the origin, reach zero in finite time $T*$ which depends on (x^0, y^0). Moreover the optimal control $\hat{u}(t)$ does not have a limit at $t \to T - 0$.
- There exist optimal solutions of the model problem that represent logarithmic spirals:

$$x^*(t) = B_1 t^2 e^{i\alpha \ln|T^*-t|},\ y^*(t) = B_2 t e^{i\alpha \ln|T^*-t|},$$
$$u^*(t) = -e^{i\alpha \ln|T^*-t|},$$
$$\alpha = \pm\sqrt{5},\ B_1 = \frac{1}{126}(4+i\alpha)(3+i\alpha),$$
$$B_2 = \frac{1}{126}(4+i\alpha)(3+i\alpha)(2+i\alpha)$$

As $t \to T^*$ the control $u^*(t)$ makes countably many rotations along the circle S^1 in finite time, $x^*(t), y^*(t) \to 0$ and switches to a singular mode $x = y = 0$.

Spherical Inverted Pendulum



B : point mass m, S : movable base of mass M

B is attached to a rigid massless rod of length ℓ

x_1: angle between SB and $O\eta\zeta$,

x_2 : angle between SB and $O\xi\zeta$

(ξ, η) : position of S, (u_1, u_2): the control forces

Control problem for linearized model

$$\ddot{x}_1 = \frac{M+m}{ml} g x_1 - \frac{1}{Ml} u_1, \qquad \ddot{x}_2 = \frac{M+m}{ml} g x_2 - \frac{1}{Ml} u_2$$

$$\int_0^\infty (x_1^2(t) + x_2^2(t))\, dt \to \min$$

$$\dot{x} = y, \quad \dot{y} = Kx + u,$$

$$x(0) = x^0, \quad y(0) = y^0.$$

Here $x, \quad y, \quad u \in \mathbb{R}^2$, K is a 2×2 diagonal matrix, $K = diag\,\{k_1, k_2\}$.
The control force is bounded:

$$u_1^2 + u_2^2 \leq 1$$

Main results for sperical pendulum

In a sufficiently small neighborhood of the origin there exist solutions of (10)–(20) that attain the upper equilibrium position and have the form of logarithmic spirals

$$x(t) = C_x(T-t)^2 e^{i\varrho ln|T-t|}(1 + g_x(T-t)),$$
$$y(t) = C_y(T-t) e^{i\varrho ln|T-t|}(1 + g_y(T-t)),$$
$$u(t) = -C_u e^{i\varrho ln|T-t|}(1 + g_u(T-t)),$$

Here $0 < T < \infty$ is a time at which solution hits the origin (the hitting time), $g_{x,y,u}(T-t) \to 0$ *as* $t \to T$, $\varrho > 0$, $C_{x,y,u} \in \mathbb{C}$.

References

THANK YOU !